

On some generalizations of Newton non-degeneracy for hypersurface singularities

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ABSTRACT. We introduce two generalizations of Newton non-degenerate singularities of hypersurfaces. Roughly speaking, an isolated hypersurface singularity is called *topologically Newton non-degenerate* if the local embedded topological singularity type can be restored from a *collection* of Newton diagrams (for some coordinate choices). A singularity that is not topologically Newton non-degenerate is called *essentially Newton-degenerate*. For plane curves we give an explicit characterization of topologically Newton non-degenerate singularities; for hypersurfaces we provide several examples.

Next, we treat the question: whether Newton non-degenerate or topologically Newton non-degenerate is a property of singularity types or of particular representatives. Namely, is the non-degeneracy preserved in an equi-singular family? This fact is proved for curves. For hypersurfaces we give an example of a Newton non-degenerate hypersurface whose equi-singular deformation consists of essentially Newton-degenerate hypersurfaces.

Finally, we define the *directionally Newton non-degenerate germs*, a subclass of topologically Newton non-degenerate ones. For such singularities the classical formulas for the Milnor number and the zeta function of the Newton non-degenerate hypersurface are generalized.

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1. INTRODUCTION

We work with germs of complex algebraic (or locally analytic) hypersurfaces in \mathbb{C}^n , mostly with isolated singularities. By the singularity type we always mean the local embedded topological type of a hypersurface germ. For the standard notions of singularity theory see [AGLV-book] and [GLS-book].

1.1. To every germ of singular hypersurface (with fixed local analytic coordinates) the Newton diagram Γ_f is associated. A germ $V_f = \{f = 0\} \subset (\mathbb{C}^n, 0)$ is called Newton non-degenerate (or non-degenerate with respect to its Newton diagram Γ_f) if for each face $\sigma \in \Gamma_f$, the truncation f_σ of f to σ is non-degenerate (i.e. the corresponding hypersurface has no singular points in the torus $(\mathbb{C}^*)^n$). A germ is called *generalized Newton non-degenerate* if it is Newton non-degenerate for some choice of coordinates.

The Newton diagram of a Newton non-degenerate germ is a *complete* invariant of the singularity type of the germ. Namely, if $(V_f, 0)$ and $(V_g, 0)$ are two Newton non-degenerate germs, such that $\Gamma_f = \Gamma_g$ then they have the same embedded topological type [Oka79]. This distinguishes the generalized Newton non-degenerate germs as especially simple to deal with. For them many topological invariants of the singularity can be expressed explicitly (or at least estimated) via the geometry of the Newton diagram in a relatively simple manner. For example:

- the Milnor number [Kouchnirenko76] (cf. also [GLS-book, I.2.1, pg.122])
- the modality (with respect to right equivalence) for functions of two variables (conjectured in [Arnol'd74, 9.9], proved in [Kouchnirenko76, Proposition 7.2])

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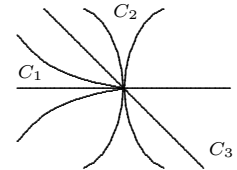
Part of the work was done in Mathematisches Forschungsinstitut Oberwolfach, during the author's stay as an OWL-fellow. Some of the results were published in the preprint [Kerner-OWP].

- the zeta function of monodromy [Varchenko76] (cf. also [AGLV-book, II.3.12])
- the spectrum [Steenbrink76, Khovanski-Varchenko85] (cf. also [Kulikov98, II.8.5])
- the Hodge numbers $h^{p,q}$ [Danilov-Khovanski87]
- the Bernstein-Saito polynomial [BGMM89]
- a bound on the Lojasiewicz invariant [Abderrahmane05] and the Lojasiewicz-type inequalities on the sufficiency [Fukui91].

Unfortunately, the condition to be generalized Newton non-degenerate is very restrictive, even in the case of plane curves.

Example 1.1. For the germ $(C, 0) \subset (\mathbb{C}^2, 0)$ consider the *tangential decomposition*: $C = \bigcup_{i=1}^k C_\alpha$.

Here each C_α has unique tangent line l_i (but may contain several branches), and the lines l_i are all distinct. Therefore, the equation of the tangent cone T_C is $\{l_1^{p_1} \cdots l_k^{p_k} = 0\}$, where p_i is the multiplicity of $(C_\alpha, 0)$, hence $\sum p_i = p$ is the multiplicity of $(C, 0)$. For example, for an ordinary multiple point $p_1 = \dots = p_k = 1$. Note that $p_i = 1$ iff C_α is a smooth branch, not tangent to any other. The point is that if $(C, 0)$ is a generalized Newton non-degenerate germ then $p_i > 1$ for *at most* two i 's.



Indeed, first notice that (in some local coordinates) if $(C, 0)$ is non-degenerate, then each $(C_\alpha, 0)$ is non-degenerate. Moreover, for some fixed i , if $p_i > 1$ and C_α is non-degenerate with respect to its diagram, then a coordinate axis must be tangent to C_α (to reflect the fact that some monomials are absent). Hence, in general, there are “not enough coordinate axes” to encode the singularity. In particular, many singularities with small Milnor numbers and quite simple defining equations are not generalized Newton non-degenerate. For example, the union of three cuspidal branches (A_2) with pairwise distinct tangents are so.

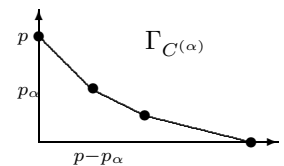
On the other hand, a locally irreducible plane curve singularity (branch) is generalized Newton non-degenerate if and only if it has only one Puiseux pair. An example of not generalized Newton non-degenerate branch is $(x^2 + y^3)^2 + x^2 y^{3+m}$, $m > 0$ with $\mu = 15 + 2m$ (named as $W_{1,2m}^\sharp$ in [AGLV-book, §I.2.3]).

1.2. Motivation. The present work has originated from the observation that many germs of curves are “almost” generalized Newton non-degenerate. Namely, for many of them the singularity type (and thus many of their properties) is reflected on Newton diagram, one just has to take *several choices* of coordinates.

Example 1.2. Continue the previous example, and consider again tangential decomposition $C = \bigcup_{\alpha=1}^k C_\alpha$. For each

$1 \leq \alpha \leq k$, let $C^{(\alpha)}$ be a germ of curve with the tangential decomposition: $C^{(\alpha)} = \left(\bigcup_{j=1}^{p-p_\alpha} L_j \right) \cup C_\alpha$. Here $\{L_j\}$ are some lines, such that any two are distinct and none is tangent to C_α (but arbitrary otherwise). Call such a germ: *the directional approximation* of $(C, 0)$. (The germ is non-unique, but its topological singularity type is unique and any two such approximations are connected by a μ -constant family.)

If C_α is generalized Newton non-degenerate then so is each $C^{(\alpha)}$, and the singularity type of C_α can be restored from some Newton diagram of $C^{(\alpha)}$ (cf. the picture, where the first segment has slope -1 , the remaining part ‘is’ the diagram of C_α). Since this diagram is the diagram of the original germ $(C, 0)$ too, the singularity type of $(C, 0)$ is completely determined from the *collection* of Newton diagrams (corresponding to all the directional approximations). The precise statement is in §3.3.



Note that if at least one branch of the curve is not generalized Newton non-degenerate then no choice of coordinates can help to recognize the singularity type in such a way.

1.3. Results. In this work we address some natural questions arising from these examples.

- Let $(V_f, 0) \subset (\mathbb{C}^n, 0)$ be an isolated hypersurface-germ. Suppose all of its Newton diagrams are known, for any choice of coordinates. *Which properties of the germ are determined by this information?*

It turns out, e.g. that the collection of all possible diagrams (each labeled with the local coordinate system which determines it) fix the projectivized tangent cone completely. Namely, if $f = f_p + f_{p+1} + \dots$ is the Taylor expansion (in some fixed coordinate system), then f_p is determined up to scaling from the collection of all diagrams (Proposition 2.1). In fact, even more information can be retrieved (cf. §2), e.g., if f_p is reduced and $n > 2$, then f_{p+1} is also fixed (up to multiplication of f by an invertible germ).

- *What are the germs whose embedded topological type is completely determined by all the possible Newton diagrams?* We call such germs: *topologically Newton non-degenerate* singularities (the name was suggested by E. Shustin). The

precise definition is in §3. It is not clear currently, how broad this class is (in particular all the generalized Newton non-degenerate germs are such), or how to classify such germs.

For plane curves ($n = 2$) we give the complete classification in §3.3: a germ is topologically Newton non-degenerate iff each branch of it is generalized Newton non-degenerate and the union of any two branches is generalized Newton non-degenerate.

For hypersurfaces ($n > 2$) the situation is much more complicated. We give some examples of topologically Newton non-degenerate germs in §3. For example all the singularities of Yomdin type ($f_p + f_{p+k}$ with f_p reduced and f_{p+k} generic and the ideal (f_p, f_{p+k}) is radical) are such. Germs that are not topologically Newton non-degenerate are called *essentially Newton-degenerate*.

• *Is topologically Newton non-degenerate (or generalized Newton non-degenerate) a property of the singularity type or of the germ?* Namely, suppose an embedded topological type has a topologically Newton non-degenerate (or Newton non-degenerate) realization. Is the generic realization of the type topologically Newton non-degenerate (or generalized Newton non-degenerate)? Or, is this notion preserved in a μ -constant deformation?

This is true for quasi-homogeneous singularities (when the Newton diagram is constant along the μ -constant stratum) by [Varchenko82], cf. also [AGV-book, III.14.3 Theorem 8]. For Newton non-degenerate case the Newton diagram can change essentially along the μ -constant stratum [Briançon-Speder75]. In [Altmann91] some cohomological conditions on the constancy of Newton diagram are given.

Our question can be considered as a weakening of the properties above. The answer is *yes* for the case of curves (Corollary 3.7) and *no* for higher dimensions. We give examples in §3.4 of μ^* -constant surface families $(S_t, 0) \subset (\mathbb{C}^3, 0) \times \mathbb{C}^1$ with the central fibre $(S_0, 0)$ Newton non-degenerate, while the generic fibre not generalized Newton non-degenerate (or even not topologically Newton non-degenerate). In fact this situation is typical.

An immediate consequence is the comparison of the equisingular strata versus the ND-topological strata. Recall that for a fixed Newton non-degenerate representative $(f^{-1}(0), 0) \subset (\mathbb{C}^n, 0)$ of a given singularity type, the ND-topological stratum is defined as the collection of all the hypersurfaces which in some coordinates have the diagram Γ_f and are non-degenerate with respect to Γ_f . For quasi-homogeneous singularities the ND-topological and μ -constant strata coincide [Varchenko82]. As our examples show, in many cases the ND-topological strata have positive codimension inside the equisingular strata.

• Once a germ is proven to be topologically Newton non-degenerate, its singularity type is determined by the associated collection of Newton diagrams. Therefore, every topological singularity invariant can be expressed (at least theoretically) via the combinatorics of the diagrams. However, as the class of topologically Newton non-degenerate singularities is so broad, it seems difficult to do this generally. Rather, we restrict to a subclass of *directionally Newton non-degenerate* singularities (introduced in §4). These are germs with Newton non-degenerate directional approximation (the natural generalization of the case of curves, example 1.2). The number of Newton diagrams needed to determine the singularity type is bounded in this case. For example, if the projectivized tangent cone $\mathbb{P}T_{(V_f, 0)}$ of the hypersurface has only isolated singularities, then the singularity type of $(V_f, 0)$ can be determined by $|Sing(\mathbb{P}T_{(V_f, 0)})|$ coordinate choices.

For directionally Newton non-degenerate singularities we generalize some classical formulas. In particular in §5 the formulas for the Milnor number and for the zeta function of monodromy are generalized. For curves we generalize also the formula for modality.

1.4. Conventions and notations. In general, we work in the space of the locally analytic hypersurface germs in \mathbb{C}^n . Sometimes we pass to the space of germs of (high) bounded degrees (to have a finite dimensional space, to use algebraicity and Zariski topology). As the singularities are isolated this is always possible by finite determinacy.

Note that for u a locally invertible function: $\Gamma_f = \Gamma_{uf}$. Hence the Newton diagram is well defined by the zero set V_f . The Newton diagrams are assumed to be comode or convenient (Γ_f intersects all the coordinate axes), unless explicitly stated. Denote by f_σ the restriction of the function f to the face $\sigma \in \Gamma_f$. Denote by Γ_f^- the set of real points on or below Γ_f .

Throughout the paper we use equisingular deformations and μ -constant deformations. In most cases the two notions coincide: for $n \neq 3$ this follows by [Lê-Ramanujam76], for $n = 3$ and deformations f_t linear in t it follows by [Parusinski99]. In fact our μ -constant deformations are often even μ^* -constant.

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2. WHAT IS DETERMINED BY THE COLLECTION OF NEWTON DIAGRAMS?

In this section $(V_f, 0)$ and $(V_g, 0) \subset (\mathbb{C}^n, 0)$ are germs of isolated hypersurface singularities, such that for *any choice* of coordinates $\Gamma_f = \Gamma_g$. Let $f = f_p + f_{p+1} + \dots$ and $g = g_p + g_{p+1} + \dots$ be the Taylor expansions in some fixed local coordinate system. For any $k \geq 0$, let $I_k(f) := \langle f_p, f_{p+1}, \dots, f_{p+k} \rangle \subset \mathcal{O} := \mathcal{O}_{(\mathbb{C}^n, 0)}$ be the ideal generated by the corresponding homogeneous forms. Finally, let $\text{Rad}(I)$ denote the radical of the ideal I .

Proposition 2.1. *Under the above assumptions one has:*

- (1) *For any $k \geq 0$ the radicals of ideals coincide: $\text{Rad}(I_k(f)) = \text{Rad}(I_k(g))$. In particular, for $k > 0$, $\mathcal{O}/\text{Rad}(I_{k-1}(f)) \supset \text{Rad}(\langle [f_{p+k}] \rangle) = \text{Rad}(\langle [g_{p+k}] \rangle) \subset \mathcal{O}/\text{Rad}(I_{k-1}(g))$.*
- (2) *Moreover, regarding the tangent cone, one has $f_p = g_p$, up to multiplication by a constant.*
- (3) *For a fixed $k > 0$, suppose that $I_i(f)$ is radical for any $0 \leq i \leq k$. Then $I_k(f) = I_k(g)$ and $\sum_{i=0}^k f_{p+i} = \sum_{i=0}^k (a_i + b_i(x)g_{p+i})$, where $a_i \in \mathbb{C}^*$ and $b_i(x) : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ for all i .*

Proof. (1) Consider the projective hypersurfaces $\{\mathbb{P}V_{f_{p+i}} \subset \mathbb{P}^{n-1}\}_{0 \leq i \leq k}$. Suppose $x \in \bigcap_{i=0}^k \mathbb{P}V_{f_{p+i}}$. By $GL(n)$ action, one can assume that $x = [0, \dots, 0, 1]$. Thus, none of the equations f_{p+i} contains the monomial x_n^{p+i} . In particular, the diagram Γ_f intersects the x_n -axis at a point higher than $(p+k)$. By the assumption on f and g , the same fact holds for Γ_g . Therefore $x \in \bigcap_{i=0}^k \mathbb{P}V_{g_{p+i}}$. Hence, (set theoretically) $\left(\bigcap_{i=0}^k \mathbb{P}V_{f_{p+i}}\right)_{\text{red}} = \left(\bigcap_{i=0}^k \mathbb{P}V_{g_{p+i}}\right)_{\text{red}}$, and by Nullstellensatz (1) follows.

(2) Write $f_p = \prod_{i=1}^l f_{p,i}^{n_i}$ and $g_p = \prod_{i=1}^l g_{p,i}^{m_i}$ for the prime decompositions of f and g , where $f_{p,i} = g_{p,i}$ up to scaling, cf. part (1). Let $x \in V_{f_{p,i}}$ be a generic point, so that x is a smooth point of the reduced cone $V_{\prod f_{p,i}}$. Apply linear transformation ϕ to $(\mathbb{C}^n, 0)$ in order to get $x = [1, 0, \dots, 0]$. Then the monomial $x_1^{\deg(f_{p,i})}$ does not appear in $f_{p,i}$, while for any $j \neq i$ the monomial $x_1^{\deg(f_{p,j})}$ does appear in $f_{p,j}$. Thus, the number $p - \deg(f_{p,i})n_{p,i}$ can be restored from the Newton diagram $\Gamma_{\phi(f)}$ by checking the monomial containing the highest power of x_1 . Hence, by equality of the Newton diagrams, one gets $n_i = m_i$. This shows that the scheme structure of the projectivized tangent cone is also restored from the collection of the Newton diagrams.

(3) The coincidence of the ideals is proved by induction. First, note that $\langle f_p \rangle = \langle g_p \rangle$. Suppose that for all $0 \leq i \leq k$ the ideals $I_i(f)$ are radical and $I_{k-1}(f) = I_{k-1}(g)$. Hence, by (1) and induction, $I_k(f) = \text{Rad}(I_k(f)) = \text{Rad}(I_k(g)) = \text{Rad}(\langle f_p, \dots, f_{p+k-1}, g_{p+k} \rangle)$. Hence $g_{p+k} = \alpha f_{p+k} + \sum_{i < k} \beta_i f_{p+i}$. Using grading one gets that α is a (zero or non-zero) constant. Analyzing both cases, $I_k(f) = I_k(g)$ follows. ■

Remark 2.2. The possible attempts to strengthen/generalize the above proposition are obstructed:

(1) It is important to take radicals (for the first statement), as the following example shows. Suppose $f_p = x_1^p$ with $p \geq 3$ and f_{p+1} contains the monomial x_2^{p+1} . Then all the relevant coordinate transformations have the form $x_1 \mapsto x_1 + Q(x_1, \dots, x_n)$ with Q quadratic, and the other coordinates (x_1, \dots, x_n) are moved by a linear transformation of $GL(\mathbb{C}_{x_2, \dots, x_n}^{n-1})$ (which preserves the hyperplane $x_1 = 0$). Write f_{p+1} as $\tilde{f}_{p+1}(x_2, \dots, x_n) + x_1 \tilde{f}_p$. Then the possible Newton diagrams fix the homogeneous form \tilde{f}_{p+1} completely, but impose no restrictions on \tilde{f}_p . E.g., $f = x_1^p + x_2^{p+1}$ and $g = x_1^p + x_1 x_2^p + x_2^{p+1}$ have equal Newton diagrams in any coordinates, but of course $I_1(f) \neq I_1(g)$.

(1') Similarly, the property $\mathcal{O}/\text{Rad}(I_{k-1}(f)) \supset \text{Rad}(\langle [f_{p+k}] \rangle) = \text{Rad}(\langle [g_{p+k}] \rangle) \subset \mathcal{O}/\text{Rad}(I_{k-1}(g))$ does not hold without the radicals, as it is exemplified by the plane curves defined by the polynomials $f = x^2 y^2 + x^5 + y^5$ and $g = x^2 y^2 + x^5 - y^5$. Note that the corresponding Newton diagrams coincide in any coordinate system. (By the direct check, a change of coordinates whose linear part is identity has no influence on the diagrams.) But, clearly, $\mathcal{O}/\text{Rad}(\langle [f_4] \rangle) \supset \langle [f_5] \rangle \neq \langle [g_5] \rangle \subset \mathcal{O}/\text{Rad}(\langle [g_4] \rangle)$.

(2) It is not possible to consider the filtration determined by the Newton diagram instead of Taylor expansion. Indeed, even in the case of a quasi-homogeneous filtration, the lowest order parts do not necessarily coincide. As an example, consider $f(x, y, z) = (z^3 + x^4 y^5)^r + x^{8r} + y^{10r} + \tilde{f}$ and $g(x, y, z) = z^{3r} + x^{4r} y^{5r} + x^{8r} + y^{10r} + \tilde{f}$. Here $r \geq 2$ and \tilde{f} consists of monomials above the hyperplane $\frac{z}{3r} + \frac{x}{8r} + \frac{y}{10r} = 1$. Note that f, g are Newton non-degenerate and for any coordinate system $\Gamma_f = \Gamma_g$. This last statement can be verified as follows. Take a locally analytic transformation ϕ of $(\mathbb{C}^3, 0)$. If its linear part mixes the coordinates (i.e. if it is not diagonal or a permutation) then obviously

$\Gamma_{\phi^*(f)} = \Gamma_{\phi^*(g)}$. Hence, one can assume that the linear part is the identity. Analyzing the non-linear parts, all the relevant cases are $z \mapsto z + \varphi$, where $\varphi \in \langle x^2, xy, xy^2, y^2, y^3 \rangle$. By a direct verification, one gets again $\Gamma_{\phi^*(f)} = \Gamma_{\phi^*(g)}$. But the lowest order parts of f, g differ significantly.

(3) (Regarding the last statement of the proposition.) If I_k is radical then I_0 is such i.e. f_p is reduced. However, in general, the fact that f_p is reduced seems to be not enough to prove that the other intermediate ideals are reduced as well (at least not by general theory of ideals). Below is an example, communicated to me by D. Eisenbud and B. Ulrich to whom I am very grateful, which shows the subtlety of the problem.

Consider the ideal in $\mathbb{C}[a, b, c, d, e, f, g]$ generated by the 3 minors of the matrix $\begin{pmatrix} a^2 & d^3 & f^4 \\ bc & e^3 & g^4 \end{pmatrix}$. The 3 equations f_5, f_6, f_7 have degrees 5, 6, 7. The total ideal is reduced (this can be checked using e.g. [GPS-Singular]). But the ideal generated by the first two equations is non-reduced, e.g. it contains $a^2 f_7$ but not $a f_7$.

Note that this example is general, it does not use all the assumptions of the proposition above. In particular the singularity of this example is non-isolated. It would be interesting to give a counterexample which fit exactly to our situation.

Corollary 2.3. With the assumptions of proposition 2.1 one has:

- (1) Let $f = f_p + f_{p+k} + \dots$ be the Taylor expansion and suppose that the ideal $\langle f_p, f_{p+k} \rangle$ is radical. Then g is contact equivalent to $f_p + f_{p+k} + \text{some higher order terms}$ (which might be different from those of f).
- (2) Let $f = f_p + f_{p+k} + f_{p+q} + \dots$ with $q > k$ be the Taylor expansion and suppose that the ideal $\langle f_p, f_{p+k}, f_{p+q} \rangle$ is radical. Then g is contact equivalent to $f_p + f_{p+k} + \lambda f_{p+q} + \text{some higher order terms}$, for some $\lambda \in \mathbb{C}$.

Proof. We prove the second claim, the first is proved similarly. The last part of proposition 2.1 gives: $g = a f_p + b f_{p+k} + c f_{p+q} + \dots$ for a, b, c locally analytic and locally invertible. Hence there exists a locally analytic coordinate scaling: $(x_1, \dots, x_n) \rightarrow \delta(x_1, \dots, x_n)$ such that $\frac{a}{\delta^p} = \frac{b}{\delta^{p+k}}$. Apply it and divide g by $\frac{a}{\delta^p}$ to get: $g \sim f_p + f_{p+k} + \lambda f_{p+q} + \dots$

■

Remark 2.4. Regarding the possible converse of proposition 2.1 we note the following:

- (a) Suppose that the components of the Taylor expansion (for some choice of coordinates) satisfy: $\sum_{i=p}^{p+k} f_i = \sum_{i=p}^{p+k} g_i(a_i + \alpha_i(x))$ with $a_i \in \mathbb{C}^*$ and $\alpha_i : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$. Then the Newton diagrams (in that fixed coordinate system) coincide up to the order $p+k$ (i.e. the parts of Γ_f, Γ_g lying below the hyperplane $\sum_i x_i = p+k$ coincide).

This follows immediately from the fact that $\Gamma_{\sum_i f_i} = \Gamma_{\sum_i a_i f_i}$ for $a_i \in \mathbb{C}^*$ and $\Gamma_{\sum_i f_i} = \Gamma_{\sum_i f_i(1+\alpha_i(x))}$.

On the other hand, the equality $\sum_{i=p}^{p+k} f_i = \sum_{i=p}^{p+k} g_i(a_i + \alpha_i(x))$ in *some* coordinate system does not imply that the (truncated) Newton diagrams coincide for *any* choice of coordinates. An elementary example is: $f = xy + y^4$ and $g = xy - y^4$.

- (b) The equality of radical ideals does not imply any relation of Newton diagrams or singularity types. As an example consider $f = x^p + xy^p + y^q$ and $g = x^p + y^q$ (for $q > p+1$).

3. TOPOLOGICALLY NEWTON NON-DEGENERATE HYPERSURFACES

3.1. Preparations for the definition. Start from the following observation. Let $(V_f, 0) = \{f = 0\} \subset (\mathbb{C}^n, 0)$ be a generalized Newton non-degenerate isolated singularity. Fix some coordinates (x_1, \dots, x_n) . Let $\phi \circ (\mathbb{C}^n, 0)$ be a locally analytic coordinate change, such that $\phi^*(f)$ is non-degenerate with respect to its diagram $\Gamma_{\phi^*(f)}$. In the space of all the locally analytic series at the origin (or, in some of its truncations if it is necessary) consider the stratum:

$$(1) \quad \Sigma_{(\phi, \Gamma_{\phi^*(f)})} := \overline{\{g \in \mathbb{C}\{x\} \mid \Gamma_{\phi^*(g)} = \Gamma_{\phi^*(f)}\}}.$$

Here the closure is taken in the classical topology (for the coefficients of the defining series). This stratum is irreducible. Then for the generic point $g \in \Sigma_{(\phi, \Gamma_{\phi^*(f)})}$ the *local embedded topological types* of $(V_f, 0)$ and $(V_g, 0)$ coincide [Kouchnirenko76].

This can be rephrased as follows: any small deformation of f inside $\Sigma_{(\phi, \Gamma_{\phi^*(f)})}$ is μ -constant (cf. §1.4).

Recall the notion of Newton weight function [AGLV-book, I.3.8] associated to every commode Newton diagram. Namely, $\lambda_\Gamma : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is defined uniquely by the conditions: $\lambda_\Gamma(\alpha \vec{x}) = \alpha \lambda_\Gamma(\vec{x})$ (for any $\alpha \in \mathbb{R}_+$) and $\lambda_\Gamma(\Gamma) = 1$.

Given two diagrams we say $\Gamma_1 \geq \Gamma_2$ if $\lambda_{\Gamma_1}(\Gamma_2) \leq 1$ (or $\lambda_{\Gamma_2}(\Gamma_1) \geq 1$).

Suppose a collection of pairs $\{(\phi_i, \Gamma_i)\}$ is given (with $\phi_i \circ (\mathbb{C}^n, 0)$ local coordinate changes and Γ_i Newton diagrams).

Definition 3.1. *The stratum of hypersurfaces germs, associated with the collection $\{(\phi_i, \Gamma_i)\}$ is the closure of the set of all the germs giving the prescribed diagrams in the prescribed coordinates, i.e.*

$$(2) \quad \Sigma_{\{(\phi_i, \Gamma_i)\}} := \{g \in \mathbb{C}\{x\} \mid \Gamma_{\phi_i^*(g)} \geq \Gamma_i \text{ for all } i\}.$$

Lemma 3.2. *For any collection $\{(\phi_i, \Gamma_i)_i\}$, as above, the associated stratum $\Sigma_{\{(\phi_i, \Gamma_i)\}}$ is a (non-trivial) linear subspace of the space of all locally analytic functions at the origin. In particular it is closed, irreducible and the notion of the generic point is well defined.*

Proof. The condition $\Gamma_{\phi_i^*(g)} \geq \Gamma_i$ means the absence of some monomials in the Taylor expansion of $\phi_i^*(g)$. This says that some directional derivatives of $\phi_i^*(g)$ vanish: $\sum a_{j_1 \dots j_n} \partial_{y_1}^{j_1} \dots \partial_{y_n}^{j_n} (g \circ \phi_i) = 0$. Here $\{y_j = \phi_i(x_j)\}$ are the new coordinates. And these conditions are linear in g in any coordinate system. ■

3.2. The main definition.

Definition 3.3. *The function $f \in \mathbb{C}\{x\}$ is called topologically Newton non-degenerate if there exist a finite number of coordinate choices (i.e. locally analytic $\phi_i \hookrightarrow (\mathbb{C}^n, 0)$) and the Newton diagrams $\{\Gamma_i\}$ such that $\{\Gamma_{\phi_i^*(f)} = \Gamma_i\}$ and any small deformation of f inside the stratum $\Sigma_{\{(\phi_i, \Gamma_i)\}}$ is μ -constant.*

Recall that for a locally invertible $u \in \mathbb{C}\{x\}$ one has $\Gamma_{uf} = \Gamma_f$ and $(V_{uf}, 0) = (V_f, 0)$. Moreover, f is topologically Newton non-degenerate iff uf is such. Thus the hypersurface germ $(V, 0) \subset (\mathbb{C}^n, 0)$ is defined to be topologically Newton non-degenerate if one (and hence any) of its locally defining functions is topologically Newton non-degenerate.

The definition 3.3 is equivalent to the following: the general point of $\Sigma_{\{(\phi_i, \Gamma_i)\}}$ corresponds to a hypersurface germ, whose singularity type is that of $(V_f, 0)$. General here means: lying in the complement of a proper analytic subset. Some comments are in order. Note that the μ -constant deformation is equisingular (for $n = 3$ use the linearity of the space $\Sigma_{\{(\phi_i, \Gamma_i)\}}$ and see the remark in §1.4). Thus by semi-continuity of μ we deduce that the subset of the germs $f \in \Sigma_{\{(\phi_i, \Gamma_i)\}}$ with the given topological type is open; and for germs in the complement the Milnor number is strictly larger. So f is topologically Newton non-degenerate iff it belongs to this (Zariski) open set.

Example 3.4. (1) Every generalized Newton non-degenerate germ is topologically Newton non-degenerate. In this case, by definition, just one pair (ϕ, Γ) suffices.

- (2) Let $C = \bigcup_{i=1}^k C_\alpha$ be the tangential decomposition of a plane curve singularity (cf. example 1.1). If each of C_α is generalized Newton non-degenerate then C is topologically Newton non-degenerate. Indeed, make k choices of coordinates $(x_1^{(i)}, x_2^{(i)})$ with $\hat{x}_2^{(i)}$ axis generic and $\hat{x}_1^{(i)}$ axis chosen such that the germ C_α is Newton non-degenerate. Then get the collection of Newton diagrams similar to those of example 1.2. Obviously, this collection of diagrams specifies the singularity type uniquely (the diagram Γ_i specifies the type of C_α and the fact that the tangent line of C_α is not tangent to any other $C_{j \neq i}$).
- (3) The curve germ $(x^2 - y^3)(x^2 - y^3 + x^3) = 0$ is the union of two branches, each being Newton non-degenerate, but the union is not topologically Newton non-degenerate. It is easy to see that the family $C_{t \neq 0} = \{(x^2 - y^3)(x^2 - ty^3 + x^3) = 0\}$ has constant Newton diagram for any choice of coordinates. But $\mu(C_{t \neq 0}) < \mu(C_1)$.
- (4) (Yomdin series.) Let $f = f_p + f_{p+k} + \dots$ with f_{p+k} generic with respect to f_p . Namely, $\mathbb{P}V_f$ and $\mathbb{P}V_{p+k}$ intersect transversally in \mathbb{P}^{n-1} , in particular $\text{Sing}(\mathbb{P}V_f) \cap \mathbb{P}V_{p+k} = \emptyset$. Assume that the ideal $\langle f_p, f_{p+k} \rangle$ is radical. Then $(V_f, 0) \subset (\mathbb{C}^n, 0)$ is topologically Newton non-degenerate.

Indeed, if for some $g \in \mathbb{C}\{x\}$ the Newton diagrams Γ_g, Γ_f coincide in any coordinates, then by corollary 2.3 g is contact equivalent with a germ of the form $f_p + f_{p+k} + \text{higher order terms}$. But all these germs are topologically contact equivalent to $f_p + f_{p+k}$, hence g and f are topologically equivalent. Finally note that the set of all the coordinate systems can be replaced by a finite set (by the argument as in the proposition below).

We believe that the condition " $\langle f_p, f_{p+k} \rangle$ is radical" is unnecessary here, but do not have any rigorous proof.

Proposition 3.5. *(Consistency of the definition.)*

- (1) Let $(V_f, 0) \subset (\mathbb{C}^n, 0)$ be a topologically Newton non-degenerate germ and $\{(\phi_i, \Gamma_i)_{i=1}^k\}$ a collection of pairs fulfilling the condition of definition 3.3 (i.e. specifying the singularity type of V_f uniquely). Then for any additional pair $(\phi_{k+1}, \Gamma_{k+1})$ the collection $\{(\phi_i, \Gamma_i)_{i=1}^{k+1}\}$ also fulfills the condition of the definition.
- (2) Let $\{(\phi_i, \Gamma_i)_i\}_{i \in I}$ be an infinite collection such that the associated stratum $\Sigma_{\{(\phi_i, \Gamma_i)\}_{i \in I}}$ (defined similarly to the case of finite collection) is of positive dimension. Suppose that any small deformation of f inside this stratum is μ -constant. Then there exists a finite sub-collection $J \subset I$ such that the conditions of the definition 3.3 are satisfied for $\{(\phi_i, \Gamma_i)_{i \in J}\}$ and hence $(V_f, 0)$ is topologically Newton non-degenerate.

Proof.

- (1) By assumption any small deformation of $(V_f, 0)$ inside the stratum $\Sigma_{\{(\phi_i, \Gamma_i)_{i=1}^k\}}$ is μ -constant, hence this is true for the substratum $\Sigma_{\{(\phi_i, \Gamma_i)_{i=1}^{k+1}\}}$ as well.
- (2) By finite determinacy one can pass to finite jets J_N of some high order. Let $j_N(g)$ be the N -jet of g . Then $\mu(V_f, 0) = \mu(V_{j_N(f)}, 0)$ and any small deformation of $(V_{j_N(f)}, 0)$ inside $\Sigma_{\{(\phi_i, \Gamma_i)\}_{i \in I}} \cap J_N$ is μ -constant. Present the stratum $\Sigma_{\{(\phi_i, \Gamma_i)\}_{i \in I}} \cap J_N$ as successive intersections: $\bigcap_{i \in J} \Sigma_{(\phi_i, \Gamma_i)} \cap J_N$, where $J \subset I$ is an increasing filtration of I .

At each step we get a linear subspace of the *finite dimensional* space of N -jets. Each intersection either decreases the dimension or has no influence. Therefore the process stabilizes after a finite number of intersections. Hence, there exists a finite subset $J \subset I$ satisfying the conditions of the definition. ■

3.3. The case of curves. For curves it is possible to give a very explicit equivalent definition of topological Newton-non-degeneracy.

Proposition 3.6. *Let $(C, 0) = \cup_i (C_\alpha, 0)$ be the tangential decomposition. Then $(C, 0)$ is topologically Newton non-degenerate iff each $(C_\alpha, 0)$ is topologically Newton non-degenerate. Moreover $(C_\alpha, 0)$ is topologically Newton non-degenerate iff the following two conditions are satisfied:*

- *Each branch of C_i is generalized Newton non-degenerate, i.e. locally it is of the type $x^p + y^q$ with $(p, q) = 1$.*
- *In addition, the union of any two singular branches is a generalized Newton non-degenerate singularity. More precisely, there does not exist a pair of singular branches in C_α with local equation (in some coordinates): $(x^p + y^q + \dots)(x^p + y^q + \dots)$. Here the dots mean higher order terms (i.e. monomials lying over the Newton diagram of $x^p + y^q$).*

The last condition can be rephrased as follows: for any pair of singular branches C_i, C_j with the Puiseux pairs (p_i, q_i) and (p_j, q_j) either $(p_i, q_i) \neq (p_j, q_j)$ or $(p_i, q_i) = (p_j, q_j)$ but the intersection multiplicity $\text{mult}_0(C_i \cap C_j) \leq p_i q_i$.

Proof. For the first part note that the collection of $\{\phi_i, \Gamma_i\}$ that specifies the type of $(C, 0)$ does the same for each $(C_\alpha, 0)$ independently. Conversely, let $\cup_i \{\phi_{\alpha, i}, \Gamma_{\alpha, i}\}$ be a collection that specifies the type of $(C_\alpha, 0)$. Then the total collection $\cup_{i, \alpha} \{\phi_{\alpha, i}, \Gamma_{\alpha, i}\}$ specifies the type of $(C, 0)$.

Thus, in the sequel we assume that all the branches of $(C, 0)$ have a common tangent.

⇒ Suppose $(C, 0)$ contains a branch $(C_i, 0)$ which is not generalized Newton non-degenerate. Choose coordinates in which the defining equation of C_i can be written as: $(x^{ar} + \dots + y^{br}) + h = 0$. Here $1 < a < b$, $(a, b) = 1$, the part $(x^{ar} + \dots + y^{br})$ is quasi-homogeneous and degenerate and h is of higher order with respect to the weights above. Let $(C'_i, 0)$ be a generic germ with the Newton diagram of C_i , in particular C'_i is Newton non-degenerate. Note, that the Newton diagrams of C_α, C'_i coincide in any coordinates.

Consider the germ $C' = \bigcup_{j \neq i} C_j \cup C'_i$. By the construction C, C' have different singularity types, but their diagrams coincide in all coordinates.

The same applies to the case of $(x^p + y^q + \dots)(x^p + y^q + \dots)$: no choice of coordinates can distinguish this from the pair of branches $(x^p + y^q + \dots)(x^p - y^q + \dots)$ (which is certainly of different type).

⇐ We want to determine the singularity type of C by choosing different coordinates. For each (generalized Newton non-degenerate) branch C_i let (p_i, q_i) be its Puiseux pair (for smooth branches take $(1, \infty)$). Let ϕ_i be a choice of coordinates for which C_i is defined by $x_i^p + y_i^q + \dots = 0$. (The smooth branches are rectified, i.e. the equation becomes $x = 0$).

Let Γ_i be the Newton diagram of $(C, 0)$ in the coordinates defined by ϕ_i . Then Γ_i contains a segment of slope p_i/q_i (for smooth branches the Newton diagram is non-commode). Then the collection $\{\phi_i, \Gamma_i\}_{i=1}^r$, for r —the number of branches, suffices to determine the singularity type of $(C, 0)$.

Indeed, let $C(t) \subset \Sigma_{\{\phi_i, \Gamma_i\}_{i=1}^r}$ be a small deformation. By construction, after the change of coordinates ϕ_i , the family contains (as a subvariety) the family defined by $(x^{p_1 m} \alpha_1(t) + \dots + y^{q_1 m} \alpha_{m+1}(t)) + \dots = 0$. Here inside the terms brackets correspond to a quasi-homogeneous form, the dots outside correspond to the higher order monomials. By construction, the quasi-homogeneous form $x^{p_1 m} \alpha_1(0) + \dots + y^{q_1 m} \alpha_{m+1}(0)$ is non-degenerate with $\alpha_1(0) \neq 0$, hence the same holds for t small enough.

Thus, the family $C(t)$ can be decomposed as $\cup C_i(t)$ and the type of each branch is preserved. (Moreover, all the smooth branches of $C(0)$ stay constant.) Finally the intersection degrees of the branches are constant (fixed by the set $\{\phi_i, \Gamma_i\}$). ■

The proof of the proposition gives an upper bound for the number of the needed choices of coordinate system to restore the singularity type: the number of branches. This bound can be improved (e.g. in each coordinate system one takes both axes tangent to some branches). But, e.g. for r singular pairwise non-tangent branches one certainly needs at least $r/2$ coordinate choices.

The proposition also allows to answer positively the question from the introduction: for curves being generalized Newton non-degenerate or topologically Newton non-degenerate are properties of singularity types and not only of their representatives.

Corollary 3.7. Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be a generalized Newton non-degenerate (or topologically Newton non-degenerate) germ of curve. Let $(C', 0) \subset (\mathbb{C}^2, 0)$ be a germ of the same singularity type as $(C, 0)$. Then $(C', 0)$ is also generalized Newton non-degenerate (or topologically Newton non-degenerate).

Proof. • For the topologically Newton non-degenerate case the statement follows immediately from proposition 3.6 (as the conditions are formulated in terms of the singularity types of the branches). Next we consider the other case.

• The topological characterization of generalized Newton non-degenerate curve-germs is well known in the folklore, but we could not find any reference, except for the preprint [GaLePi07].

Another way to prove the statement is as follows. By [GLS-book, Proposition 2.17(2), pg. 287] the miniversal equisingular deformation of any (isolated) Newton non-degenerate plane curve singularity can be realized by monomials not below the Newton diagram, hence consists of Newton non-degenerate germs. Let \mathbb{S} be the singularity type of the Newton non-degenerate germ $(C, 0)$, recall that the equisingular stratum $\Sigma_{\mathbb{S}}$ of germs of the singularity type \mathbb{S} is globally irreducible. Thus we get: there exists a (Zariski) open dense subset of $\Sigma_{\mathbb{S}}$, whose points correspond to generalized Newton non-degenerate curve-germs.

On the other hand, for $(C, 0) = f^{-1}(0)$, let Γ_f be the Newton diagram in the fixed coordinates (so that f is Γ_f non-degenerate). Consider the stratum of curves that (in *some* coordinates) can be brought to Γ_f or to a bigger diagram:

$$(3) \quad \Sigma_{\Gamma_f} := \{g \in \mathbb{C}\{x\} \mid \text{for some change of coordinates } \phi \circ (\mathbb{C}^2, 0) \Gamma_{\phi^*(g)} \geq \Gamma_f\}$$

By definition the stratum is closed, its points correspond either to generalized Newton non-degenerate curve-germs of type \mathbb{S} or to higher types adjacent to \mathbb{S} . The natural morphism $\Sigma_{\Gamma_f} \rightarrow \overline{\Sigma}_{\mathbb{S}}$ is defined by $g \rightarrow (g^{-1}(0), 0)$. By the remark above this morphism is dominant, hence in fact is surjective. Which means: *every* point of $\Sigma_{\mathbb{S}}$ corresponds to a generalized Newton non-degenerate curve germ. ■

3.4. Germs vs types.

In this subsection we discuss the questions of §1.3 in higher dimensional cases.

For $n \geq 3$, being topologically Newton non-degenerate or generalized Newton non-degenerate is a property of germ representatives (or of analytic singularity types) but not of the topological types. We construct equisingular families of surfaces $(V_t, 0) \subset (\mathbb{C}^3, 0)$ such that V_0 is Newton non-degenerate in the classical sense but $V_{t \neq 0}$ is degenerate (in various senses).

Remark 3.8. Two observations are useful. Let $(V_f, 0) \subset (\mathbb{C}^3, 0)$ with fixed coordinates, such that f is non-degenerate for Γ_f . So, the coordinates in $\mathbb{P}^2 = \text{Proj}(\mathbb{C}^3)$ are fixed too.

- Suppose the projectivized tangent cone $\mathbb{P}T_{(V_f, 0)} \subset \mathbb{P}^2$ is irreducible, reduced (hence with isolated singularities). Then its singular locus lies in $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1] \in \mathbb{P}^2$. In particular $\mathbb{P}T_{(V_f, 0)} \subset \mathbb{P}^2$ can have at most three singular points.
- Let $pt \in \text{Sing}(\mathbb{P}T_{(V_0, 0)})$ and suppose the (plane curve) singularity $(\mathbb{P}T_{(V_0, 0)}, pt)$ is not an ordinary multiple point. Let $T = \{l_1^{p_1} \cdots l_k^{p_k} = 0\}$ be the tangent cone of $(\mathbb{P}T_{(V_0, 0)}, pt)$, assume $p_1 > 1$. Then (as f is Newton non-degenerate) l_1 coincides with one of the coordinate axes.

The examples below are based on two ideas: moving several mild singularities of the tangent cone (for the case of generalized Newton non-degenerate) or deforming one strong singularity of the tangent cone (for the case of topologically Newton non-degenerate).

Example 3.9. Consider a super-isolated singularity [Luengo87] $V_0 = \{f_p + f_{p+1} = 0\} \subset (\mathbb{C}^3, 0)$. Here f_{p+1} is generic and the projective curve $\{f_p = 0\} \subset \mathbb{P}^2$ has three cusps (assume p is big enough, e.g. $p \geq 6$). According to the remark above, arrange the coordinates such that the cusps are at $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1] \in \mathbb{P}^2$ (corresponding to the coordinate axes \hat{x} , \hat{y} and \hat{z} of \mathbb{C}^3).

Note that a $GL(3)$ action which fixes this points is at most a permutation. To make V_0 Newton non-degenerate assume that the tangents to the cusps are oriented along the coordinate axes, e.g. $f_p = z^{p-3}(zx^2 + y^3) + x^{p-3}(xy^2 + z^3) + y^{p-3}(yz^2 + x^3)$.

Let V_t be the equi-singular family, with the cusps staying at their points $\hat{x}, \hat{y}, \hat{z}$, but their tangents changing freely. For example, $f(t) = f_p(t) + f_{p+1}$ with $f_p(t) = z^{p-3}(z(x - ty)^2 + y^3) + x^{p-3}(x(y + tz)^2 + z^3) + y^{p-3}(y(z - tx)^2 + x^3)$.

Then $V_{t \neq 0}$ is topologically Newton non-degenerate but not generalized Newton non-degenerate. Indeed, to bring $V_{t \neq 0}$ to a Newton non-degenerate form one should keep the cusps at the points $\hat{x}, \hat{y}, \hat{z}$ and at the same time keep

their tangents along the axes. So, only $GL(3)$ transformations are relevant. But, as was noted above, the only $GL(3)$ transformations which keep the cusps at the points $\hat{x}, \hat{y}, \hat{z}$ are permutations.

Example 3.10. (cf. also [Altmann91, example 5.3]) Consider the family of surfaces $f_t = f_5(t) + f_6 = x^5 + z(zx + ty^2)^2 + y^5 + z^6$. This is a super-isolated singularity because $Sing(f_5 = 0) \cap (f_6 = 0) = \emptyset$. The projectivized tangent cone of these surface is the plane quintic $\{f_5(t) = 0\} \subset \mathbb{P}^2$ with one A_4 point at $[0, 0, 1] \in \mathbb{P}^2$. Thus $\mu = 68 = (5-1)^3 + 4$ (see §5.1 for the general formula). The family is equisingular in t , e.g. because each surface V_t is resolved by one blowup of the origin and the singularity type of exceptional divisor in the surface is independent of t .

The singularity $V_{t=0}$ is Newton non-degenerate (by direct check). On the other hand for $t \neq 0$ the singularity is not generalized Newton non-degenerate. To show this, we prove that the restriction of f_t to the face $\Gamma_{f_t} \cap Span(x^5, y^5, z^5)$ is degenerate for any choice of coordinates. Let $\phi \circ (\mathbb{C}^3, 0)$ be a local coordinate change. As we are interested in the face whose monomials correspond to the tangent cone, the non-linear part of ϕ is irrelevant. So, assume $\phi \in GL(\mathbb{C}^3)$ and acts on $\mathbb{P}T_S = \{f_5 = 0\} \subset \mathbb{P}^2$. Thus the goal is to bring the singularity of this quintic to a Newton non-degenerate form. But this is impossible for $t \neq 0$, because in local coordinates the curve is defined by $(x + ty^2)^2 + x^5 + y^5 = 0$. So, to bring it to a Newton non-degenerate form we must do a *non-linear* transformation $x \rightarrow x - ty^2$ on \mathbb{C}^2 , which does not arise from $GL(3) \circ (\mathbb{C}^3, 0)$.

Example 3.11. *Equisingular deformation to essentially Newton-degenerate singularities.* In the last example all the fibres are topologically Newton non-degenerate (by corollary 4). The next example, in which the generic fibre is essentially Newton-degenerate (i.e. not topologically Newton non-degenerate), is a simple alteration. The goal is to construct a Newton diagram with the properties: any linear change of coordinates erases the essential information of the singularity, while any coordinate change whose linear part is the identity preserves the diagram.

For this, one changes the inclinations of the face on which the degeneration occurs ($Span(x^5, z^3x^2, zy^4, y^5)$ in the last example) and adds some other faces.

Consider the hypersurface $f = x^a + y^b + z^c + z^k(zx + y^2)^2$. (For $(a, b, c, k) = (5, 5, 6, 1)$ one has the previous example.) Suppose (a, b, c, k) are such that the Newton diagram consists of the three faces (cf. the picture): $Conv(x^a, x^2z^{k+2}, y^b)$, $Conv(y^b, x^2z^{k+2}, y^4z^k)$ and $Conv(x^2z^{k+2}, y^4z^k, z^c)$. This can be ensured by next convexity conditions:

- x^a and z^c lie above the plane $Span(x^2z^{k+2}, y^4z^k, y^b)$. The equation of this plane is $\frac{y}{b} + \frac{z}{c}(1 - \frac{4}{b}) + x(\frac{2}{b} - \frac{1-\frac{4}{b}}{k}) = 1$, so the conditions are:

$$(4) \quad a(\frac{2}{b} - \frac{1-\frac{4}{b}}{k}) > 1 \text{ and } \frac{c}{k}(1 - \frac{4}{b}) > 1.$$

- z^c and y^4z^k lie above the plane $Span(x^a, x^2z^{k+2}, y^b)$. The equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{1-\frac{2}{a}}{k+2}z = 1$, so the conditions are

$$(5) \quad c > \frac{k+2}{1-\frac{2}{a}} \text{ and } \frac{k}{1-\frac{4}{b}} > \frac{k+2}{1-\frac{2}{a}}.$$

- x^a and y^b lie above the plane $Span(z^c, x^2z^{k+2}, y^4z^k)$. The equation of the plane is $\frac{z}{c} + x\frac{1-\frac{k+2}{c}}{2} + y\frac{1-\frac{k}{c}}{4} = 1$ giving:

$$(6) \quad c > \frac{k+2}{1-\frac{2}{a}} \text{ and } \frac{b}{4}(1 - \frac{k}{c}) > 1.$$

Assume further that $b < a < k+4 < c$ and also: if $\phi \circ (\mathbb{C}^2, 0)$ is any locally analytic transformation whose linear part is identity (i.e. $(x, y, z) \rightarrow (x + \phi_x, y + \phi_y, z + \phi_z)$ with $\phi_i \in m^2$) then $\Gamma_f = \Gamma_{\phi^*(f)}$. This can be achieved e.g. if for each face the angles with all the coordinate planes are bounded $\frac{1}{2} < \tan(\alpha) < 2$.

For this it is enough to assume: z^{c-1} lies below the plane $Span(x^a, x^2z^{k+2}, y^b)$.

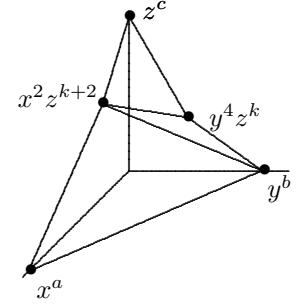
Summarizing, all the restrictions above are implied by the following inequalities:

$$(7) \quad b < \frac{k+2}{1-\frac{2}{a}} < \frac{k}{1-\frac{4}{b}} < \min(c, 2b), \quad b < a < \min(2b, k+4) < c < k+6.$$

This implies $c = k+5$ and $k > 10$. We consider (possibly the simplest case): $f_t = x^{14} + y^{13} + z^{16} + z^{11}(zx + ty^2)^2$. By direct check this family is equisingular (e.g. $\mu_t = 2220 = \text{const}$, can be calculated using [GPS-Singular]).

The generic fibre $f_t^{-1}(0)$ is essentially Newton-degenerate by the following proposition.

Lemma 3.12. *Let $f_{t \neq 0}$ as above and $g = x^{14} + y^{13} + z^{16} + z^{11}(z^2x^2 + y^4)$, Newton non-degenerate. Then $f_{t \neq 0}$ and g have the same Newton diagram in any coordinate system. But $\mu(g) = 2219 < \mu(f_t) = 2220$, so $f_{t \neq 0}$ is not topologically Newton non-degenerate.*



Proof. Let $\phi \circ (\mathbb{C}^2, 0)$ be a locally analytic change of coordinates whose linear part is identity. By the construction it preserves the Newton diagram. Therefore it is enough to consider only *linear* coordinate changes. But then only the monomials x^a, y^b are relevant and their coefficients are the same in both cases. ■

Remark 3.13.

- The families in the examples above are not just μ -constant but μ^* -constant (by direct computation). In fact, one can show that in these families all the polar multiplicities are preserved, i.e. the surfaces are c-cosecant in the sense of [Teissier77].
- In the examples above the singular types admit some Newton non-degenerate representatives, but the generic representatives are not generalized Newton non-degenerate. More precisely, consider the equisingular stratum in the space of miniversal deformation. The locus of generalized Newton non-degenerate surface-germs is of positive codimension.

4. DIRECTIONALLY NEWTON NON-DEGENERATE SINGULARITIES

As was shown above, the topologically Newton non-degenerate germs form quite a broad class, difficult to work with. One could consider other intermediate classes of singularities (between generalized Newton non-degenerate and topologically Newton non-degenerate). We study here the minimal generalization: directionally Newton non-degenerate germs. These are higher dimensional analogs of example 1.2.

For a fixed diagram Γ define the subset $\Delta \subset \Gamma$ of faces "far from the \hat{x}_n axis" as follows. Let \hat{x}_n be the unit normal to the hyperplane $\text{Span}(x_1, \dots, x_{n-1})$. For each top-dimensional face $\sigma \subset \Gamma$ let $\text{Span}(\sigma)$ be the supporting hyperplane. Let \hat{v}_σ be the unit normal to $\text{Span}(\sigma)$, oriented such that $(\hat{x}_n, \hat{v}_\sigma) > 0$.

We define:

$$(8) \quad \Delta := \overline{\cup \left\{ \sigma \subset \Gamma \atop \text{top-dimensional} \mid (\hat{x}_n, \hat{v}_\sigma) \geq \frac{1}{\sqrt{n}} \right\}}.$$

Since the union is over the top dimensional faces, Δ can be empty. Here the topological closure is needed to add the relevant non-top-dimensional faces.

Example 4.1. In the case of plane curves, with $\Gamma \subset \mathbb{R}_{\geq 0}^2$, one gets that Δ is the union of all the faces whose angle with the \hat{x}_1 is not bigger than $\pi/4$.

For the Newton diagram of the surface $x_1 x_2 x_3 + x_1^p + x_2^q + x_3^r$ with $p, q, r \geq 3$ one has: $\Delta = \text{Conv}(x_1 x_2 x_3, x_1^p, x_2^q)$.

For the Newton diagram of the surface $x_3^N + x_2^p + x_3^p$ with $N > p$ one has: $\Delta = \emptyset$.

In the sequel we assume that the projectivized tangent cone $\mathbb{P}T_{(V_f, 0)}$ has only isolated singularities, in particular it is reduced.

Definition 4.2. *The isolated hypersurface singularity $(V_f, 0) \subset (\mathbb{C}^n, 0)$ is directionally Newton non-degenerate if for each singular point $x \in \text{Sing}(\mathbb{P}T_{(V_f, 0)})$ there exists a coordinate-change $\phi \circ (\mathbb{C}^n, 0)$ such that $\phi(x) = [0, \dots, 0, 1]$ and the restriction $\phi^*(f)|_{\Gamma \setminus \Delta}$ is non-degenerate.*

Lemma 4.3. $\left(\begin{array}{c} \text{generalized} \\ \text{Newton-non-degenerate} \\ \text{with } \text{Sing}(\mathbb{P}T_{(V_f, 0)}) \text{ isolated} \end{array} \right) \Rightarrow \left(\begin{array}{c} \text{directionally} \\ \text{Newton-non-degenerate} \end{array} \right) \Rightarrow \left(\begin{array}{c} \text{topologically} \\ \text{Newton-non-degenerate} \end{array} \right).$

Proof. The first implication is obtained as follows. Suppose f is Newton non-degenerate in some fixed coordinates and $pt \in \text{Sing}(\mathbb{P}T_{(V_f, 0)}) \subset \mathbb{P}^{n-1}$. Let $p = \text{mult}(V_f, 0)$. By the non-degeneracy on the face $\Gamma_f \cap \text{Span}(x_1^p, \dots, x_n^p)$ one gets that $pt \notin (\mathbb{C}^*)^{n-1}$, so by a permutation of coordinates we can assume: $pt = [0, *, \dots, *] \in \mathbb{P}^{n-1}$. Now consider the restriction of f to the boundary $\partial(\Gamma_f \cap \text{Span}(x_1^p, \dots, x_n^p))$. Checking the non-degeneracy on the top-dimensional components one has (after a permutation) $pt = [0, 0, *, \dots, *] \in \mathbb{P}^{n-1}$. After several similar steps we get that $pt = [0, 0, \dots, 0, 1]$.

For the second implication suppose $(V_f, 0)$ is directionally Newton non-degenerate. Consider the corresponding stratum

$$(9) \quad \Sigma_f = \overline{\{g \mid \Gamma_g = \Gamma_f \text{ in all coordinate systems}\}}.$$

- First we prove that any family $(V(t), 0) \subset \Sigma_f$ for t small enough consists of directionally Newton non-degenerate hypersurfaces. Recall that the tangent cone is constant in this family (cf. proposition 2.1). For any point $pt \in \text{Sing}(\mathbb{P}T_{V(0)}) = \text{Sing}(\mathbb{P}T_{V(t)})$ choose coordinates with $pt = [0, \dots, 0, 1]$ and $f(0)|_{\Gamma \setminus \Delta}$ non-degenerate.

Note that $\Gamma_{f(t)} = \Gamma_{f(0)}$ and for the fixed Newton diagram the non-degeneracy is an open property in Zariski topology. So for each singular point $pt \in \text{Sing}(\mathbb{P}T_{V(0)})$ we have a bound: if $|t| < \epsilon(pt)$ then $f(t)|_{\overline{\Gamma \setminus \Delta}}$ is non-degenerate. Thus, as $\mathbb{P}T_{V(0)}$ has isolated singularities only, the needed bound is found by combining a finite number of inequalities.

- Now observe that such a family $(V(t), 0) \subset \Sigma_f$, for $|t| < \epsilon$, consisting of directionally Newton non-degenerate germs, is equi-resolvable in the following sense. Blowup the origin, let $\tilde{V}(t)$ be the strict transform. By construction it has a finite number of singular points $pt \in \text{Sing}(\mathbb{P}T_{V(0),0}) \subset \mathbb{P}^{n-1}$, their positions are independent of t . At each such point the germ $(\tilde{V}(t), pt_\alpha)$ is Newton non-degenerate in some local coordinates, the choice of the coordinates is independent of t , and in the chosen coordinates the Newton diagram $\Gamma_{(\tilde{V}(t), pt_\alpha)}$ is independent of t . Now we can use various nice properties of families of Newton non-degenerate hypersurfaces [Oka79].

Apply the toric modification to the multigerms $\coprod_\alpha (Bl(\mathbb{C}^n), pt_\alpha)$, corresponding to the diagrams $\Gamma_{(\tilde{V}(t), pt_\alpha)} = \Gamma_{(\tilde{V}(0), pt_\alpha)}$. Let $(Y, E) \rightarrow (\mathbb{C}^n, 0)$ be the resulting total modification of the original ambient space. Now the strict transform $\tilde{\tilde{V}}(t)$ of $(V(t), 0)$ is a family of *smooth* hypersurfaces, intersecting transversally the exceptional divisor E . Moreover, for each irreducible component $E_i \subset E$ the intersections $C(t) = \tilde{\tilde{V}}(t) \cap E_i$ form a family of embedded varieties $(E_i, C(t))$, which is locally trivial over the base $(\mathbb{C}_t^1, 0)$. In particular, for any small t , one has embedded diffeomorphism $(E_i, C(t)) \approx (E_i, C(0))$.

- Now we prove that such a family is μ -const. The Milnor number is determined by the monodromy zeta function, hence it is enough to show that the zeta function is constant. And the later is immediate by the theorem of [A'Campo75], we recall it below §5.2, equation (16). Indeed, by the above discussion, for each m we have that $\chi(S_m)$ is independent of t .

Finally, observe that μ -constant implies equisingularity, for $n = 3$ note that the family can always be chosen linear. ■

The three non-degeneracy classes are distinct as is seen already in the case of curves:

Proposition 4.4. *The plane curve germ $(C, 0) \subset (\mathbb{C}^2, 0)$ is directionally Newton non-degenerate iff in its tangential decomposition $(C, 0) = \cup (C_\alpha, 0)$ each $(C_\alpha, 0)$ is generalized Newton non-degenerate. In particular, being directionally Newton non-degenerate is a property of the singularity type. Namely, if $(C, 0)$ is directionally Newton non-degenerate then any other germ of the same type is directionally Newton non-degenerate.*

In particular, for curves being directionally Newton non-degenerate places no conditions on the tangent cone. In higher dimensions the tangent cone is more restricted.

Proposition 4.5. *Let $(V_f, 0) \subset (\mathbb{C}^n, 0)$ be directionally Newton non-degenerate.*

- (1) *Every singular point of $\mathbb{P}T_{(V_f, 0)} \subset \mathbb{P}^{n-1}$ can be brought to a Newton non-degenerate form by a linear transformation (the corresponding Newton diagram can be non-commode).*
- (2) *Let $f = f_p + f_{p+k}$ with $\text{Sing}(\mathbb{P}T_{V_f})$ isolated and f_{p+k} generic with respect to f_p (i.e. $\text{Sing}(V_{f_p}) \cap V_{f_{p+k}} = \emptyset$). If f_p (i.e. $\mathbb{P}T_{(V_f, 0)}$) satisfies the condition (1) above then $(V_f, 0)$ is directionally Newton non-degenerate.*

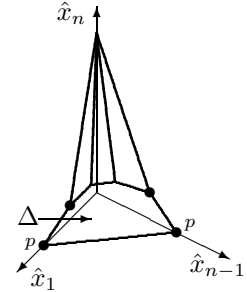
Proof.

- (1) Let $pt \in \text{Sing}(\mathbb{P}T_{V_f, 0})$, choose the coordinates such that $pt = [0, \dots, 0, 1]$ and $f|_{\overline{\Gamma \setminus \Delta}}$ is non-degenerate. Let the Taylor expansion be $f = f_p + \dots$. Note that the Newton diagram of the singularity $(f_p^{-1}(0), 0) \subset (\mathbb{P}^{n-1}, 0)$ is obtained from $\Gamma_{(V_f, 0)}$ by intersecting with the hyperplane $\sum x_i = p$ and projecting to the coordinate hyperplane $\text{Span}(x_1, \dots, x_{n-1})$. Thus, in the fixed coordinates, f_p is non-degenerate for its Newton diagram. Finally, observe that f_p was brought to this form by a $GL(n)$ transformation, because for any locally analytic change of coordinates in $(\mathbb{C}^n, 0)$ only the linear part acts nontrivially on the tangent cone.
- (2) Let $pt \in \text{Sing}(\mathbb{P}T_{V_f, 0}) \subset \mathbb{P}^{n-1}$. As the transformation, which brings $(\mathbb{P}T_{(V_f, 0)}, pt)$ to a non-degenerate form, is linear, it can be lifted to a linear transformation of $(\mathbb{C}^n, 0)$. Then, the restriction of the transformed f to the relevant boundary components of $\Gamma_f \cap \text{Span}(x_1^p, \dots, x_n^p)$ is non-degenerate. Hence, by the genericity of f_{p+k} , the restriction $f|_{\overline{\Gamma \setminus \Delta}}$ is non-degenerate. ■

4.1. The directional approximations. We want to generalize the directional approximations of curves (introduced in example 1.2). The reader is invited to review the definition of "faces far from the \hat{x}_n axis".

Definition 4.6. Let $(V, 0) \subset (\mathbb{C}^n, 0)$ be directionally Newton non-degenerate, in particular $\mathbb{P}T_{(V,0)} \subset \mathbb{P}^{n-1}$ has isolated singularities only. We say that $(V, 0)$ has a Newton non-degenerate directional approximation $(V_1, 0), \dots, (V_k, 0)$ if for any point $pt_\alpha \in \text{Sing}(\mathbb{P}T_{(V,0)})$ there exists a coordinate system $\phi_\alpha \circ (\mathbb{C}^n, 0)$, such that:

- ★ $\phi_\alpha(pt_\alpha) = [0, \dots, 0, 1] \in \mathbb{P}^{n-1}$ and the subset Δ of faces "far from the \hat{x}_n axis" lies inside the hyperplane $\sum x_i = \text{mult}(V, 0)$
- ★ $\Delta \cap \text{Span}(\hat{x}_1, \dots, \hat{x}_{n-1}) = \text{Conv}(x_1^p, \dots, x_{n-1}^p)$ for $p = \text{mult}(V, 0)$
- ★ $f|_{\Gamma_{\phi_\alpha^*(f)} \setminus \Delta}$ is Newton non-degenerate, $\Gamma_{(V_\alpha, 0)} = \Gamma_{(\phi_\alpha^*(f))}$ and $(V_\alpha, 0)$ is Newton non-degenerate.



Note that the existence of a Newton non-degenerate directional approximation, in general, imposes additional restriction for f :

Example 4.7. • Any directionally Newton non-degenerate plane curve singularity has a Newton non-degenerate directional approximation (cf. example 1.2).

• On the other hand, this is not the case if $n \geq 3$. For such n , a sufficient condition is the following. Let $(V, 0) \subset (\mathbb{C}^n, 0)$ be directionally Newton non-degenerate, with $\text{Sing}(\mathbb{P}T_{(V,0)}) = \{pt_1, \dots, pt_k\} \subset \mathbb{P}^{n-1}$. Suppose the Newton diagram of any germ $(\mathbb{P}T_{(V,0)}, pt_\alpha) \subset (\mathbb{C}^{n-1}, z_i)$ remains comode after any $GL(n-1)$ change of coordinates (preserving pt_α). Then there exists a Newton non-degenerate directional approximation.

Similarly to the case of curves, the germs $(V_i, 0)$ are not defined uniquely. But any two representatives (associated with a fixed point p_α) have the same singularity type, the same Newton diagram and can be joined by a μ -constant family.

5. SOME NUMERICAL SINGULARITY INVARIANTS

Here we generalize some classical theorems to directionally Newton non-degenerate germs which admit Newton non-degenerate directional approximation (cf. definition 4.6). The proofs of the statements are in §5.4 and §5.6.

5.1. Kouchnirenko's formula for the Milnor number. Recall, that for a Newton non-degenerate germ the Milnor number is determined by the Newton diagram by the classical Kouchnirenko formula:

$$(10) \quad \mu(V, 0) = \sum_{j=0}^n (-1)^{n-j} j! \text{Vol}_j(\Gamma_{(V,0)}),$$

where $\text{Vol}_j(\Gamma_{(V,0)})$ is the sum of j -dimensional volumes of the intersections of Γ with the j -dimensional coordinate hyperplanes. The volume of unit cube is 1 in any dimension.

Theorem 5.1. Let $(V, 0) \subset (\mathbb{C}^n, 0)$ be an isolated directionally Newton non-degenerate hypersurface singularity with multiplicity p . Let $\text{Sing}(\mathbb{P}T_{(V,0)}) = \{pt_1, \dots, pt_k\}$ and let $(V_1, 0), \dots, (V_k, 0)$ be its Newton non-degenerate directional approximations (cf. §4.1). Then Kouchnirenko's formula holds in the following form:

$$(11) \quad \mu(V, 0) = \sum_{\alpha=1}^k \mu(V_\alpha, 0) - (k-1)(p-1)^n, \text{ where } \mu(V_\alpha, 0) \text{ is computed by the formula (10).}$$

Example 5.2. • For plane curves this proposition is just a reformulation of tangential decomposition, see §5.4. Let $(C, 0) = \bigcup_{\alpha=1}^k (C_\alpha, 0)$ be the tangential decomposition, let μ_α, p_α be the Milnor number and the multiplicity of C_α , so that $p = \sum p_\alpha$. Then, for the directional approximation one has: $\mu(C^{(\alpha)}) = \mu_\alpha + (p - p_\alpha)(p + p_\alpha - 2)$. Thus, by (11), the total Milnor number is: $\mu = \sum \mu_\alpha + p^2 + 1 - k - \sum p_\alpha^2$.

• Let $(V_f, 0) \subset (\mathbb{C}^n, 0)$ for $f = f_p + f_{p+1} + \dots \in \mathbb{C}\{x\}$ be directionally Newton non-degenerate with the directional approximation $\{(V_\alpha, 0)\}$ corresponding to the singular points $\{pt_\alpha \in \mathbb{P}T_{(V,0)}\}$. Assume each V_α has the singularity type of the germ $\{f_p^{(\alpha)}(x_1, \dots, x_n) + x_n^{p+q_\alpha} = 0\} \subset \mathbb{C}^n$, where the hypersurface $\{f_p^{(\alpha)}(x_1, \dots, x_n) = 0\} \subset \mathbb{P}^{n-1}$ is singular at the point $[0, \dots, 0, 1] \in \mathbb{P}^{n-1}$ only. For example, directionally Newton non-degenerate singularities of Yomdin type $(f_p + f_{p+k})$ are of this kind. By the theorem the computation of $\mu(V_f, 0)$ reduces to $\mu(V_\alpha, 0)$.

Blowup \mathbb{C}^n at the origin, let \tilde{V}_α be the strict transform, then (see [Melle00] or the proof of this theorem):

$$(12) \quad \mu(V_\alpha, 0) = (p-1)^n + \mu(\mathbb{P}T_{(V_\alpha,0)}, pt_\alpha) + \mu(\tilde{V}_\alpha).$$

Note that \tilde{V}_α is the suspension (in some local coordinates its equation is of the form: $g(x_2, \dots, x_{n-1}) + x_n^{q_\alpha} = 0$) thus $\mu(\tilde{V}_\alpha) = \mu(\mathbb{P}T_{(V_\alpha,0)}, pt_\alpha)(q_\alpha - 1)$ and we obtain:

$$(13) \quad \mu(V, 0) = (p-1)^n + \sum q_\alpha \mu(\mathbb{P}T_{(V,0)}, pt_\alpha).$$

If all the q_α are equal this reproduces the result of [Luengo-Melle95] (in the directionally Newton non-degenerate case).

• It is instructive to compute some case explicitly. For example, consider $f = f_4 + f_6 + f_8$, where

$$(14) \quad f_4 = xyz(x + y + z), \quad f_6 = (x^2y^2 + y^2z^2 + x^2z^2)(x + 2y + 3z)^2, \quad f_8 = x^8 + y^8 + z^8$$

Then the curve $f_4^{-1}(0) \subset \mathbb{P}^2$ (four lines) has nodes at the points $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, $[1, 1, 0]$, $[1, 0, 1]$, $[0, 1, 1]$. The directional approximations at $[1, 1, 0]$, $[1, 0, 1]$, $[0, 1, 1]$ are of the singularity type $x^4 + y^4 + x^2z^2 + y^2z^2 + z^6$. So $q_\alpha = 2$ and $\mathbb{P}T_{(V_\alpha, 0)}$ has only one singular point: the node, so $\mu(\mathbb{P}T_{(V_\alpha, 0)}) = 1$. The directional approximations at $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$ have the singularity type of $x^4 + y^4 + x^2z^2 + y^2z^2 + z^8$. So $q_\alpha = 4$, while $\mathbb{P}T_{(V_\alpha, 0)}$ has again only one node, $\mu(\mathbb{P}T_{(V_\alpha, 0)}) = 1$.

In total one has:

$$(15) \quad \mu = 3^3 + (2 \times 1 + 2 \times 1 + 2 \times 1) + (4 \times 1 + 4 \times 1 + 4 \times 1) = 45$$

which can be of course recomputed directly, e.g. by [GPS-Singular].

5.2. Zeta function of monodromy. Recall the basic result of [A'Campo75] (see also [AGLV-book, II.3.12]).

Given an isolated hypersurface singularity, construct its good embedded resolution (see the diagram): \tilde{V} is smooth, E consists of smooth components and $\tilde{V} \cup E$ is a normal crossing divisor. Let $\pi^{-1}(0) = \sum m_i E_i$, i.e. E_i is an irreducible component of E of multiplicity m_i . Define $S_m := \{x \in E^{reg} : \text{mult}(E, x) = m\}$ (where E^{reg} is the regular part of E). Then

$$\begin{array}{ccc} (\tilde{V}, \tilde{V} \cap E) & \subset & (Y, E) \\ \downarrow & & \downarrow \pi \\ (V, 0) & \subset & (\mathbb{C}^n, 0) \end{array}$$

$$(16) \quad \zeta_{(V, 0)}(z) = \prod_{m \geq 1} (1 - z^m)^{\chi(S_m)}$$

where χ is the Euler characteristic.

The product structure of this formula is the basic reason for the possibility to determine the zeta function by the geometry of the Newton diagram.

Proposition 5.3. *Let $(V, 0) \subset (\mathbb{C}^n, 0)$ be a directionally Newton non-degenerate germ, whose projectivized tangent cone has isolated singularities: $\text{Sing}(\mathbb{P}T_V) = \{pt_1, \dots, pt_k\}$, let $(V_1, 0), \dots, (V_k, 0)$ be the corresponding directional approximations. Then A'Campo's formula can be written in the form*

$$(17) \quad \zeta_{(V, 0)}(z) = \frac{\prod_{\alpha=1}^k \zeta_{(V_\alpha, 0)}(z)}{(1 - z^p)^{(k-1)(n - \chi_{p, n-1})}}$$

where $\zeta_{(V_\alpha, 0)}(z)$ is the classical zeta-function of the (Newton non-degenerate) hypersurface-germ, $p = \text{mult}(V, 0)$ and $\chi_{p, n-1} = \chi(V_{p, n-1}) = \frac{(1-p)^{n-1}}{p} + n$ is the topological Euler characteristic of an arbitrary smooth hypersurface $V_{p, n-1} \subset \mathbb{P}^{n-1}$ of degree p .

For a related result, see [Gusein-Zade Luengo Melle-Hernandez97].

Using the last proposition it is immediate to generalize Varchenko's formula for the zeta function in terms of the Newton diagram. Recall ([Varchenko76], [AGLV-book, II.3.12]) that for an isolated Newton non-degenerate hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ the zeta function of the monodromy can be written in the form:

$$(18) \quad \zeta_{(V, 0)}(z) = \prod_{l=1}^n (\zeta^l(z))^{(-1)^{l-1}},$$

where $\{\zeta^l(z)\}_l$ are some polynomials completely determined by the geometry of l -dimensional faces of the Newton diagram.

Corollary 5.4. Under the assumptions of proposition 5.3 Varchenko's formula is valid in the following form:

$$(19) \quad \zeta_{(V, 0)}(z) = \frac{1}{(1 - z^p)^{(k-1)(n - \chi_{p, n-1})}} \prod_{pt_\alpha \in \text{Sing}(\mathbb{P}T_{(V, 0)})} \prod_{l=1}^n (\zeta_{(V_\alpha, 0)}^l(z))^{(-1)^{l-1}}$$

where, $\chi_{p, n-1} = \chi(V_{p, n-1})$ is the topological Euler characteristic of an arbitrary smooth hypersurface $V_{p, n-1} \subset \mathbb{P}^{n-1}$ of degree p , and $\zeta_{(V_\alpha, 0)}^l(z)$ are the standard Varchenko polynomials (for the Newton non-degenerate singularities V_α).

Example 5.5. • Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be the union of k non-tangent branches of types $x^{p_\alpha} + y^{q_\alpha}$ for $p_\alpha < q_\alpha$, $(p_\alpha, q_\alpha) = 1$. Let $p = \text{mult}(C, 0) = \sum p_\alpha$. The proposition 5.3 gives: $\zeta = \frac{\prod \zeta_\alpha}{(1-z^p)^{(k-1)(2-p)}}$ where $\zeta_\alpha = \frac{1-z^{p+q_\alpha-p_\alpha}}{(1-z^p)^{p-p_\alpha-1}(1-z^{p_\alpha(p+q_\alpha-p_\alpha)})}$. Altogether one has: $\zeta = (1-z^p)^{2-k} \prod_\alpha \frac{1-z^{p+q_\alpha-p_\alpha}}{1-z^{p_\alpha(p+q_\alpha-p_\alpha)}}$.

• As in the example 5.2, consider a directionally Newton non-degenerate hypersurface

$$(20) \quad (V_f, 0) \subset (\mathbb{C}^n, 0) \text{ for } f = f_p + f_{p+1} + \dots \in \mathbb{C}\{x\}$$

with the directional approximation $\{(V_\alpha, 0)\}$ corresponding to the singular points $\{pt_\alpha \in \mathbb{P}T_{(V,0)}\}$. Assume each V_α has the singularity type of the germ $\{\sum_{i < n} x_i^p + x_n^{p-p_\alpha} \sum_{i < n} x_i^{p_\alpha} + x_n^{p+q_\alpha} = 0\} \subset \mathbb{C}^n$, i.e. its projectivized tangent cone has only one singular point: an ordinary multiple point of multiplicity p_α . For simplicity assume $(p_\alpha, q_\alpha) = 1$.

Then, for $n = 3$, an immediate application of Varchenko's formula gives:

$$(21) \quad \zeta(V_\alpha, 0) = \frac{(1-z^p)^{p^2-p_\alpha^2}(1-z^{p_\alpha(q_\alpha+p)})^{p_\alpha}(1-z^{(q_\alpha+p)})(1-z^p)^2}{(1-z^p)^{2(p-p_\alpha)}(1-z^{p_\alpha(q_\alpha+p)})^2(1-z^p)^p} = (1-z^p)^{p^2-3p+3-(p_\alpha-1)^2}(1-z^{p_\alpha(q_\alpha+p)})^{p_\alpha-2}(1-z^{(q_\alpha+p)})$$

Substituting this into the formula of proposition 5.3 one has:

$$(22) \quad \zeta_{(V_f, 0)} = (1-z^p)^{p^2-3p+3-\sum_\alpha (p_\alpha-1)^2} \prod_\alpha (1-z^{(q_\alpha+p)})(1-z^{p_\alpha(q_\alpha+p)})^{p_\alpha-2}$$

As a consistency check we can get the Milnor number, as the degree of the polynomial $\frac{\zeta_{(V_f, 0)}}{1-z}$. From the last formula the degree is:

$$(23) \quad p(p^2-3p+3)-1-\sum p(p_\alpha-1)^2+\sum_\alpha \left((q_\alpha+p)+p_\alpha(q_\alpha+p)(p_\alpha-2) \right) = (p-1)^3 + \sum q_\alpha(p_\alpha-1)^2$$

recovering the result of example 5.2 for our case.

5.3. Order of determinacy. Suppose a directionally Newton non-degenerate hypersurface germ $\{f=0\} = (V, 0) \subset (\mathbb{C}^n, 0)$ has a Newton non-degenerate directional approximation $(V_1, 0), \dots, (V_k, 0)$. For each $(V_\alpha, 0)$ let $o.d.(V_\alpha, 0)$ be the (contact, topological) order of determinacy [GLS-book, I.2.2]. It is easily read from the diagram of $(V_\alpha, 0)$.

Proposition 5.6. *The order of determinacy of $(V, 0)$ is $\max_\alpha(o.d.(V_\alpha, 0))$.*

Proof. Let $q = \max_\alpha(o.d.(V_\alpha, 0))$. Suppose $\text{jet}_q(f) = \text{jet}_q(g)$, then $\Gamma_f = \Gamma_g$ in any coordinates. Hence g is topologically Newton non-degenerate with respect to its collection of the diagrams. So, the singularity types of f, g coincide.

On the other hand the order of determinacy of $(V, 0)$ is certainly at least $\max_\alpha(o.d.(V_\alpha, 0))$. ■

5.4. Proofs. The case of curves. For curves the proofs are especially simple (and they will be related with some other invariants as well, e.g. with the modality).

5.4.1. Proof of Proposition 5.1. It is based on the formulas for the δ invariant:

- $\mu = 2\delta - r + 1$, here r is the number of branches
- $\delta = \sum_\alpha \delta(C_\alpha) + \sum_{\alpha < \beta} \langle C_\alpha, C_\beta \rangle$ (for the tangential decomposition $C = \bigcup_{\alpha=1}^k C_\alpha$).

Using these formulas one gets:

$$(24) \quad \mu(C) = \sum_\alpha (\mu(C_\alpha) + r_\alpha - 1) + 2 \sum_{\alpha < \beta} \langle C_\alpha, C_\beta \rangle - \sum_\alpha r_\alpha + 1 = \sum_\alpha \mu(C_\alpha) + 1 - k + \sum_{\alpha \neq \beta} p_\alpha p_\beta$$

(for $p_\alpha = \text{mult}(C_\alpha)$). Assume that the curve-germ has Newton non-degenerate directional approximations $\{C^{(\alpha)}\}$ (i.e. each C_α is generalized Newton non-degenerate). Then the result follows from the observation: $\mu(C^{(\alpha)}) = \mu(C_\alpha) + p^2 - p_\alpha^2 + 2p_\alpha - 2p$.

5.4.2. Proof of Proposition 5.3. Let $C = \bigcup_{\alpha=1}^k C_\alpha$ be the tangential decomposition, then the resolution tree consists of pE and k chains corresponding to $\{C_\alpha\}$. Let $\{C^{(\alpha)}\}$ be the directional approximations (cf. example 1.2). The embedded resolution graph of $(C, 0)$ is obtained as the union of those of $\{C^{(\alpha)}\}$, they all have the same component pE . So, the product in equation 16 splits into parts corresponding to $\{C^{(\alpha)}\}$ and one should divide by the over-counting:

$$(25) \quad \zeta_C(z) = \frac{\prod_{\alpha} \zeta_{C^{(\alpha)}}(z)}{(1-z^p)^{(k-1)(2-p)}}$$

Note that in the case of curves this result is valid without assumption that each C_α is generalized Newton non-degenerate.

5.4.3. Corollary 5.4 can be stated very explicitly in the directionally Newton non-degenerate case.

Assume each C_α is generalized Newton non-degenerate. To write Varchenko's formula introduce the parameters of the Newton diagram of $\Gamma_{C^{(\alpha)}}$. For each edge l_j of the diagram (except for $(0, p), (p - p_\alpha, p_\alpha)$) let $a_j x + b_j y = c_j$ be the equation of the line it spans. Here $a_j, b_j, c_j \in \mathbb{N}$ and $(a_j, b_j) = 1$ (so the coefficient are fixed uniquely). Let $|l_j|$ be the number of integral points minus one, on the edge j . Then the formula of corollary 5.4 reads:

$$(26) \quad \zeta_C(z) = \frac{1}{(1-z^p)^{k-2}} \prod_{\alpha=1}^k \frac{1-z^{q_\alpha}}{\prod_{\alpha_\alpha} (1-z^{c_{\alpha_\alpha}})^{|l_{\alpha_\alpha}|}}$$

Here α_i runs over the edges of $\Gamma_{C^{(i)}}$, each time omitting the edge $(0, p), (p - p_i, p_i)$.

5.5. Right modality for functions of two variables.

For the Newton non-degenerate singularities the (right) modality can be calculated as the number of integral points (x, y) not above the Newton diagram, satisfying $x, y \geq 2$ (cf. [Kouchnirenko76]). This generalizes naturally to directionally Newton non-degenerate case.

Let $C = \bigcup_{\alpha} C_\alpha$ be the tangential decomposition, assume each C_α is generalized Newton non-degenerate. Let $\Gamma_{C^{(\alpha)}}$ be the Newton diagram of the corresponding directional approximation. The closed set $\Gamma_{C^{(\alpha)}}^-$ of points not above the Newton diagram decomposes naturally into the triangle $x + y \leq p$ for $p = \text{mult}(C)$ and the remaining polygon Δ_α . Note that Δ_α is 'half-open': if $(x, y) \in \Delta_\alpha$ then $x + y > p$.

Proposition 5.7. • In the notations above, the right modality of $(C, 0)$ is $\binom{p-2}{2} + \sum_{\alpha} \#|\Delta_\alpha \cap \mathbb{Z}_{\geq 2}^2|$, where $\#$ counts the number of lattice points.

• Alternatively, if $(\mu_\alpha, p_\alpha, \text{modality}_\alpha)$ are invariants of C_α then $\text{modality}(C) = \sum \text{modality}_\alpha + \frac{p^2}{2} - \sum \frac{p_\alpha^2}{2} + 3 - 3k$.

In the last formula the convention is: the right modality of A_0 (i.e. of a smooth branch) is 1.

Proof. By definition, the right modality is $\mu - \tau^{es}$, where τ^{es} is the codimension of the equi-singular stratum in the space of miniversal deformation. The additivity of μ for the diagrams of directional approximation follows from proposition 5.1.

For τ^{es} use the classical formula (in terms of the multiplicities of the strict transform on the resolution), cf. [GLS-book, pg. 373, eq. 2.8.36]

$$(27) \quad \tau^{es} = \sum \binom{m_q + 1}{2} - 1 - \# \left(\begin{array}{c} \text{free} \\ \text{points} \end{array} \right).$$

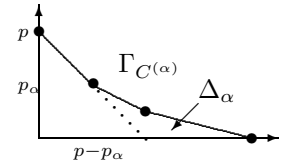
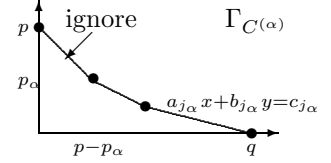
Here the sum is over the infinitely near points of $(C, 0)$ arising during the resolution. An infinitely near point of $(C, 0)$ is free if it lies on at most one exceptional divisor. In particular the initial singular point of $(C, 0)$ is free. For a smooth branch the convention is $\tau^{es}(A_0) = -1$.

Finally, as in the proof for zeta function, relate the resolution tree of $(C, 0)$ to the trees of its directional approximations $(C^{(\alpha)}, 0)$.

The second claim follows similarly using: $\tau^{es} = \sum_{\alpha} \tau^{es}(C_\alpha) + \binom{p+1}{2} - \sum \binom{p_\alpha+1}{2} + 2k - 2$. Note the convention in this formula: $\tau^{es}(A_0) = -1$. ■

Example 5.8. Let $(C, 0)$ be the union of $\{A_{n_i}\}_{i=1}^k$ with tangent lines pairwise distinct. Then for each directional approximation $\#|\Delta_\alpha \cap \mathbb{Z}_{\geq 2}^2| = 0$. Therefore, the right modality is $\binom{2k-2}{2}$.

5.6. Proofs. The case of hypersurfaces.



5.6.1. Proof of Theorem 5.1. As the Milnor number is determined by the zeta function of monodromy (since $\deg \zeta = 1 + (-1)^{n-1}\mu$), the formula can be immediately obtained from the proposition 5.3. We give also a direct proof.

The derivation of the formula is based on the following result [Melle00, Theorem 1]. For the hypersurface germ $(V, 0) \subset (\mathbb{C}^n, 0)$, let $\mathbb{P}T_{(V,0)} \subset \mathbb{P}^{n-1}$ be the projectivization of its tangent cone and $\tilde{V} \rightarrow V$ the strict transform under the blow-up of the origin. Assume, both \tilde{V} and $\mathbb{P}T_{(V,0)}$ have isolated singularities only, and set $p = \text{mult}(V, 0)$. Then:

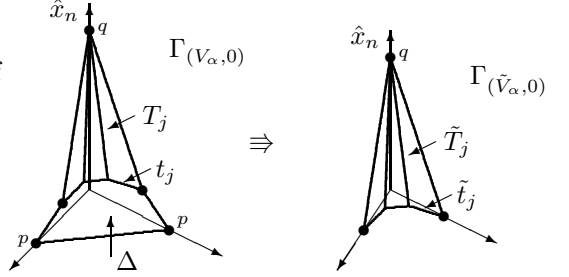
$$(28) \quad \mu(V, 0) = (p-1)^n + \mu(\mathbb{P}^{n-1}, \mathbb{P}T_{(V,0)}) + \mu(\text{Bl}_0 \mathbb{C}^n, \tilde{V}).$$

As a preparation consider the change of the Newton diagram of V_α under the blowup of $(\mathbb{C}^n, 0)$.

As in §4.1, the top-dimensional faces of the diagram are the “face far from \hat{x}_n ”, denoted Δ , and a collection $\{T_j\}$. The intersections $T_j \cap \Delta = t_j$ are faces of dimension $\leq (n-2)$. Consider the strict transform of V_α under the blowup of the origin $\{(x_1, \dots, x_n) = [\sigma_1, \dots, \sigma_n]\} \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$. (The two Newton diagrams are on the right.) The relevant chart is $\sigma_n \neq 0$, with the coordinates $(\frac{\sigma_1}{\sigma_n}, \dots, \frac{\sigma_{n-1}}{\sigma_n}, x_n)$. The total transform of the function is:

$$f(x_1, \dots, x_n) \rightarrow x_n^p \left(f_p\left(\frac{\sigma_1}{\sigma_n}, \dots, \frac{\sigma_{n-1}}{\sigma_n}, 1\right) + x_n f_{p+1}\left(\frac{\sigma_1}{\sigma_n}, \dots, \frac{\sigma_{n-1}}{\sigma_n}, 1\right) + \dots + x_n^{q-p} \right) + \dots$$

The set of points under $\Gamma_{(V_\alpha, 0)}$ is naturally subdivided into two parts, one being the pyramid under $\text{Span}(x_1^p, \dots, x_n^p)$. Denote the other part (under $\bigcup_j T_j$ but not under $\text{Span}(x_1^p, \dots, x_n^p)$) by \square_{V_α} . Let $\tilde{\square}_{V_\alpha}$ be the polyhedron under $\Gamma_{(\tilde{V}_\alpha, 0)}$.



The blowup induces the map $\square_{V_\alpha} \rightarrow \tilde{\square}_{V_\alpha}$ by $(a_1, \dots, a_n) \rightarrow (a_1, \dots, a_{n-1}, \sum_{i=1}^n a_i - p)$. (Note that by construction $\sum a_i \geq p$.) This map is an element of $SL(2, \mathbb{Z})$. So, there is a natural correspondence between the faces of \square_{V_α} and $\tilde{\square}_{V_\alpha}$.

While, \square_{V_α} and $\tilde{\square}_{V_\alpha}$ are not equal as polyhedra, their corresponding faces have equal volumes. Therefore $\mu(\square_{V_\alpha}) = \mu(\tilde{\square}_{V_\alpha})$. Here $\mu(\square_{V_\alpha})$ is the standard expression: the main volume $n! \text{Vol}_n(\square_{V_\alpha})$ minus the volume of top-dimensional faces $(n-1)! \text{Vol}_{n-1}(\square_{V_\alpha})$, plus the sum of volumes of faces of codimension 2, etc.

Similarly, let $\square_{V_\alpha}^b = \square_{V_\alpha} \cap \text{Span}(x_1^p, \dots, x_n^p)$ and $\tilde{\square}_{V_\alpha}^b = \tilde{\square}_{V_\alpha} \cap \{x_n = 0\}$. Then the corresponding faces of $\square_{V_\alpha}^b$ and $\tilde{\square}_{V_\alpha}^b$ have equal volume.

Therefore the classical Kouchnirenko formula gives (recall that \tilde{V}_α and $\mathbb{P}T_{(V_\alpha, 0)}$ have Newton non-degenerate singularities):

$$(29) \quad \mu(\square_{V_\alpha}) + \mu(\square_{V_\alpha}^b) = \mu(\tilde{\square}_{V_\alpha}) + \mu(\tilde{\square}_{V_\alpha}^b) = \mu(\tilde{V}_\alpha) + \mu(\mathbb{P}T_{(V_\alpha, 0)})$$

Finally,

- note that $\mu(\tilde{V}_\alpha) = \mu(\tilde{V}, pt_\alpha)$ and $\mu(\mathbb{P}T_{(V_\alpha, 0)}) = \mu(\mathbb{P}T_{(V, 0)}, pt_\alpha)$
- sum over all the singular point of the projectivized tangent cone $\mathbb{P}T_{(V, 0)}$ (i.e. sum over all the directional approximations V_α) and apply equation (28)
- note that each $\mu(V_\alpha, 0)$ contains a contribution from the basic pyramid (x_1^p, \dots, x_n^p) , for which $\mu = (p-1)^n$. ■

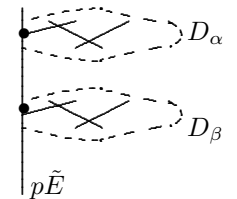
5.6.2. Proof of Proposition 5.3. Blowup \mathbb{C}^n at the origin. By the assumption the strict transform \tilde{V} of V has isolated singularities only (e.g. $(\tilde{V}, y_1), \dots, (\tilde{V}, y_k)$) and the exceptional divisor is pE for $E \approx \mathbb{P}^{n-1} \subset \text{Bl}_0(\mathbb{C}^n)$. Now, resolve the singularities of \tilde{V} .

Write the total preimage of the origin in the form: $\pi^{-1}(0) = p\tilde{E} + \sum_{\alpha=1}^k D_\alpha$. Here each D_α corresponds to the resolution of (\tilde{V}, y_α) (cf. the picture). In particular: $D_\alpha \cap D_\beta = \emptyset$ for $\alpha \neq \beta$.

Thus the product $\prod_{m \geq 1}$ in the original formula (16) can be replaced by k copies (for each directional approximation V_α). Each such copy contributes the unnecessary factor $(1 - z^p)^{\chi(\tilde{E} \setminus (D_\alpha \cup \tilde{V}_\alpha))}$ and no copy contains the needed factor $(1 - z^p)^{\chi(\tilde{E} \setminus (\bigcup_\alpha D_\alpha \cup \tilde{V}))}$.

Hence, the formula can be written in the form

$$(30) \quad \zeta_{(V, 0)}(z) = (1 - z^p)^{\chi(\tilde{E} \setminus (\bigcup_\alpha D_\alpha \cup \tilde{V}))} \cdot \prod_{\alpha=1}^k \frac{\zeta_{(V_\alpha, 0)}(z)}{(1 - z^p)^{\chi(\tilde{E} \setminus (D_\alpha \cup \tilde{V}_\alpha))}}$$



Note that (set-theoretically) $\tilde{E} \setminus (\bigcup_{\alpha} D_{\alpha} \cup \tilde{V}) = E \setminus \mathbb{P}T_{(V,0)}$ and $\tilde{E} \setminus (D_{\alpha} \cup \tilde{V}_{\alpha}) = E \setminus \mathbb{P}T_{(V_{\alpha},0)}$, so the correction factor is:

$$(31) \quad \frac{(1-z^p)^{\chi(E \setminus \mathbb{P}T_V)}}{\prod_{\alpha} (1-z^p)^{\chi(E \setminus \mathbb{P}T_{V_{\alpha}})}} = \frac{(1-z^p)^{(1-k)\chi(\mathbb{P}^{n-1})}}{(1-z^p)^{\chi(\mathbb{P}T_V) - \sum \chi(\mathbb{P}T_{V_{\alpha}})}}$$

Finally, as $\mathbb{P}T_{(V_{\alpha},0)}$ has isolated singularities only, $\chi(\mathbb{P}T_{V_{\alpha}}) = \chi(V_{p,n-1}) + N_{\alpha}$, where $V_{p,n-1} \subset \mathbb{P}^{n-1}$ is a smooth hypersurface of degree p (in particular its Euler characteristic is independent of the hypersurface) and N_{α} is a number completely determined by the local singularity types of $Sing(\mathbb{P}T_{(V_{\alpha},0)})$. Thus $\chi(\mathbb{P}T_V) - \sum \chi(\mathbb{P}T_{V_{\alpha}}) = (1-k)\chi(V_{p,n-1})$, proving the statement. ■

5.7. Some open questions.

- The right modality for hypersurfaces. Already for surfaces there exist equisingular deformations involving monomials below the Newton diagram [Altmann91]. Hence a natural generalization of §5.5 can possibly give only a lower bound.

A candidate can be guessed from the case of Brieskorn-Pham singularities as follows. For the hypersurface $\{\sum x_i^{a_i} = 0\} \subset (\mathbb{C}^n, 0)$ the miniversal equisingular deformation is spanned by monomials of the type:

$$(32) \quad \{x_1^{k_1} \cdots x_n^{k_n} \mid k_i \leq a_i - 2, \sum \frac{k_i}{a_i} \geq 1\}.$$

Apply to \mathbb{Z}^n the reflection: $(k_1, \dots, k_n) \rightarrow (a_1 - k_1, \dots, a_n - k_n)$. Then the relevant monomials are mapped to the region:

$$(33) \quad \{x_1^{k_1} \cdots x_n^{k_n} \mid 2 \leq k_i \leq a_i, \text{ the point } (\frac{k_1}{n-1}, \dots, \frac{k_n}{n-1}) \text{ is not above } \Gamma\}.$$

Note that for $n = 2$ we obtain precisely the region of Kouchnirenko's formula 5.4. Thus the natural guess is that for any n and for any Newton non-degenerate hypersurface whose Newton diagram intersects the axes at the points a_1, \dots, a_n , the right modality is at least the number of these monomials. One can easily check that this holds for suspensions of curves and T_{pqr} singularities.

- Recall that by [Saito88], for surfaces (i.e. $n = 3$) the geometric genus of a Newton-non-degenerate isolated singularity equals the number of $\mathbb{Z}_{>0}^n$ not above Γ . Can this formula be generalized to the directionally Newton non-degenerate case?

- For a topologically Newton non-degenerate singularity $f^{-1}(0) \subset (\mathbb{C}^n, 0)$ it is interesting to study the change of the *analytical* type in the stratum

$$(34) \quad \Sigma_f := \{g \in \mathbb{C}\{x\} \mid \Gamma_f = \Gamma_g \text{ in any coordinate system}\}.$$

While possible such g are very restricted, e.g. the tangent cone is constant along this stratum, some analytic invariants (e.g. the Tjurina number) can vary, see e.g. the normal family of E_{12} : $x^3 + y^7 + axy^5$. A natural question is: which analytic invariants are preserved in such families?

- Can one distinguish the spectra of Newton non-degenerate, directionally Newton non-degenerate, topologically Newton non-degenerate from the spectra of other singularities?

- Give an algorithm to check that a given hypersurface is topologically Newton non-degenerate (or directionally Newton non-degenerate, or Newton non-degenerate in some coordinates).

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